Stat 155 Lecture 2 Notes

Daniel Raban

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1 Nim and Rim

1.1 Nim

Here is a combinatorial game called Nim. We have k piles of chips, and each turn, a player removes some (positive) number of chips from some pile. The player wins when they take the last chip. Nim is an impartial combinatorial game with positions

$$X = \{(n_1, \dots, n_k) : n_i \geq 0\}.$$

The set of moves is

$$\left\{(x, y) \in X^2 : \text{some } i \text{ has } y_i < x_i,\, y_j = x_j \;\forall j \neq i\right\}.$$

The terminal position is 0, and the game follows normal play. We can think of a position $(x_1, \dots, x_i, 0, \dots, 0)$ as the position $(x_i, \dots, x_i)$ in a smaller game. So we could instead define

$$X = \{(n_1, \dots, n_k) : k \geq 1, n_i \geq 0\},$$

letting k be a part of the position. Nim is progressively bounded because from $x \in X$, there can be no more than $\sum_i x_i$ moves until the terminal position.

Example 1.1. Which positions are in N or P? $0 \in P$, but $n_1 \in N$. Also, $(1, 1) \in P$, and $(1, 2) \in N$. If $n_1 \neq n_2$, then $(n_1, n_2) \in N$; but $(n_1, n_1) \in P$.

To find the winning positions of Nim, we make the following definition.

Definition 1.1. Given a Nim position $(x_1, \dots, x_k)$, the *Nim-sum* $x_1 \oplus \cdots \oplus x_k$ is defined as follows. Write $x_1, \dots, x_k$ in binary, and add the digits in each place modulo 2; then interpret the result as the binary representation of a number.

$$\begin{array}{c|ccccl} 6 & 0 & 1 & 1 & 0 & \\ 12 & 1 & 1 & 0 & 0 & \\ 13 & 1 & 1 & 0 & 1 & \\ \hline & 0 & 1 & 1 & 1 & = 7 \end{array}$$

Example 1.2. You can check your work with these examples to see if you understand how to get the Nim-sum of a position.

1. If $x = 7$, x has Nim-sum is 7.
2. If $x = (2, 2)$, x has Nim-sum 0.
3. If $x = (2, 3)$, it has Nim-sum 1.
4. If $x = (1, 2, 3)$, it has Nim-sum 0.

Theorem 1.1 (Bouton)**.** *The Nim position $(x_1, \ldots, x_k)$ is in P iff the Nim-sum of its components is 0.*

Proof. Let $Z = \{(x_1, \ldots, x_k) : x_1 \oplus \cdots \oplus x_k = 0\}$. We will show that

1. Every move from X leads to a position outside Z.
2. For every position outsize Z, there is a move to Z, which implies that terminal positions are in Z.

From this, it will follow that $Z = P$ (exercise).

To prove 1, note that removing chips from one pile only changes one row when computing the Nim-sum. So then some place in the binary representation of the Nim-sum is changed, making it nonzero.

To prove 2, let j be the position of the leftmost 1 in the binary representation of the Nim-sum $s = x_1 \oplus \cdots \oplus x_k$. There is an odd number of $i \in \{1, 2, \ldots, k\}$ with 1 in column j. Choose one such i. Now we replace x_i by $x_i \oplus s$. That is, we make the move

$$(x_1, \ldots, x_k) \to (x_1, \ldots, x_{i-1}, x_i \oplus s, x_{i+2}, \ldots, x_k).$$

[insert picture] This decreases the value of x_i, so it is a legal move. This also changes every 1 in the binary representation of the Nim-sum to 0, making the Nim-sum 0. $\square$

1.2 Rim

Here is a game called Rim. Each position is a finite set of points in the plane and a finite set of continuous, non-intersecting loops, each passing though at least one point. Each turn, a player adds another loop. This game is progressively bounded.

Proposition 1.1. *Rim is equivalent to Nim, in the sense that we can define a mapping $\phi : X \to X_{Nim}$ such that $P = \{x \in X : \phi(x) \in P_{Nim}\}$.*

Proof. For a position x, define $\phi(x) = (n_1, \ldots, n_k)$, where the n_i are the number of points in the interiors of the connected regions bounded by the loops. This allows all of the standard Nim moves; by drawing a loop (not containing any points in its interior) that passes through some number of points in a connected component, the corresponding chips are removed. It also allows some nonstandard moves, such as moves that create more piles.

Why is $P = \{x \in X : \phi(x) \in P_{\text{Nim}}\}$? $\phi(x) = 0$ for terminal x, and some move from N leads to P; this is true because all of the standard Nim moves are available as Rim moves. We now want to show that every move from P leads to N; we need only check that if $\phi(x)$ has Nim-sum zero, then any move to $\phi(y)$ has a nonzero Nim-sum. We know this is true for a standard Nim move, so we need only check that this is true when the pile that was diminished is split. Suppose we split x_i into u and v, using up some of the vertices from x_i. We have $x_i > u + v \geq u \oplus v$. So the move changes to a nonzero Nim-sum. $\square$